

# Time reversal in photoacoustic tomography and levitation in a cavity

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## Abstract

A class of photoacoustic acquisition geometries in  $\mathbb{R}^n$  is considered such that the spherical mean transform admits an exact filtered back projection reconstruction formula. The reconstruction is interpreted as a time reversion mirror that reproduces exactly an arbitrary source distribution in the cavity. A series of examples of non-uniqueness of the inverse potential problem is constructed basing on the same geometrical technique.

## 1 Introduction

Reconstruction of a function from its spherical means is a mathematical tool for photo- and thermo-acoustic tomography and SAR technic. Typically the integral data is available for all spheres centered at a set  $Z \subset \mathbb{R}^n$  (e. g. an array of transducers) and an unknown function is supported by an open set  $H \subset \mathbb{R}^n \setminus Z$ . Closed reconstruction formulae of filtered back projection type are known for few types of central sets  $Z$ . We state here that such a reconstruction is possible for a class of algebraic central sets  $Z$  called oscillatory §§2-4. Next, we show that the reconstructions can be interpreted as application of an universal time reversal method for the wave equation. A signal is recorded by an array of transducers (mirror), time-reversed and retransmitted into the medium. The retransmitted signal propagates back through the same medium and refocuses on the source. The time reversal method is an effective tool in acoustical imaging even if the array of transducers is aside of a radiation source and only a limit angle data are available (see [8]). If arrays completely surround the support of source distribution then complete angular data can be detected.

We show that for a free space and any oscillatory array set  $Z$ , the time reversal is a perfect mirror providing the exact reconstruction of any source distribution supported by the cavity  $H$  §5. The cavity can be open; the method works for a paraboloid, a half-space and a two sheet hyperboloid giving an approximate reconstruction with a finite array of transducers. A strange point

is that the time reversal method looking quite natural in our problem works accurate only for very special class of center sets  $Z$ .

The method of oscillatory geometry can be applied also for a generalization of famous Newton's attraction theorem (Principia, 1687). This theorem states that a mass uniformly distributed over a thin sphere exerts zero gravitation field inside (levitation). This is an example of the non-uniqueness of the interior inverse potential problem. For a survey of inverse problems of potential theory [17]. We show that there are many sets  $Z$  in  $\mathbb{R}^3$  supporting a mass distribution which generate levitation in an open set §§7-10.

## 2 Oscillatory sets

**Definition.** Let  $p$  be a real polynomial in  $\mathbb{R}^n$  of degree  $m$  with the zero set  $Z$ . We call  $p$  and  $Z$  *oscillatory* with respect to a point  $a \in \mathbb{R}^n \setminus Z$ , if  $p$  has  $m$  simple zeros in  $L$  for almost any line  $L \subset \mathbb{R}^n$  through  $a$ .

**Theorem 1** *Let  $p$  be a polynomial in  $\mathbb{R}^n$ ,  $n > 1$  with a compact oscillatory zero set  $Z$  with respect to a point  $a$ . We have  $Z = Z_1 \cup \dots \cup Z_\mu$ , where  $Z_1, \dots, Z_\mu$ ,  $\mu = m/2$  are ovals (homeomorphic images of a  $n-1$  sphere). They are nested in the sense that set of regular points of  $Z_i$  is contained in the interior of  $Z_{i+1}$  for  $i = 1, \dots, \mu$ . Moreover,  $Z_1 = \partial H$  where  $H$  is the set of all hyperbolic points. It is a convex component of  $\mathbb{R}^n \setminus Z$ .*

**Proof.** Let  $a$  be a hyperbolic point of  $p$  and  $S$  be a sphere in  $\mathbb{R}^n$  with the center at the origin. For any  $\omega \in S$  we numerate zero points of  $p$  by  $x_k(\omega) = a + t_k\omega$ ,  $k = 1, \dots, \mu(\omega)$  counting with multiplicity in such a way that

$$t_{-\sigma(\omega)} \leq \dots \leq t_{-2} \leq t_{-1} < 0 < t_1 \leq t_2 \leq \dots \leq t_{\tau(\omega)}. \quad (1)$$

By Rouché's theorem for an arbitrary  $\omega_0 \in S$  and arbitrary  $u < v$ , the number of zeros  $t = t_k(\omega)$  of  $p(a + t\omega)$  such that  $u < t < v$  is constant for all  $\omega$  in a neighborhood of  $\omega_0$  if  $p(a + u\omega_0)p(a + v\omega_0) \neq 0$ . Moreover,  $t_k(\omega)$  is a  $1/m$ -Hölder continuous function for any  $k$ . This implies  $\sigma(\omega) + \tau(\omega) = m$  for any  $\omega$  since  $Z$  is compact. Because of the sphere  $S$  is connected, we have  $\sigma = \tau = \mu \div m/2$  and  $t_{-k}(\omega) = t_k(-\omega)$ ,  $k = 1, \dots, \mu$  for  $\omega \in S$ . For any  $k = 1, \dots, \mu$ , the function  $x_k(\omega) = a + t_k(\omega)\omega$  is defined and continuous for  $\omega \in S$ , hence  $Z_k = \{x = x_k(\omega), \omega \in S\}$  is an oval. We have  $Z = \bigcup_1^\mu Z_k$  and the variety  $Z_k \cap Z_j$  has dimension  $< n-1$  for any  $j \neq k$ . Therefore the sets  $Z_k$  are nested and  $a$  belongs to the interior  $H$  of the oval  $Z_1$ . It is easy to see that  $Z$  is oscillatory with respect to an arbitrary  $b \in H$  and  $H$  is convex. We omit a detailed proof which is geometrically transparent. ►

**Definition.** We call a *hyperbolic cavity* of an oscillatory set  $Z$  any maximal connected set  $H$  of points  $a$  such that  $Z$  is oscillatory with respect to  $a$ . By Theorem 1 there is only one hyperbolic cavity, if the zero set is compact.

**Examples: 1.** Two sheet hyperboloid is oscillatory with two hyperbolic cavities, whereas any ellipsoid, elliptic paraboloid, elliptic and parabolic cylin-

der is oscillatory with only one hyperbolic cavity. A slab has three hyperbolic cavities. Other second order polynomials are not oscillatory.

**2.** The zero set of the polynomial  $p \doteq (x^2 + y^2)^3 - 12(x^2 + y^2)^2 + 7x^2y^2 + 30(x^2 + y^2) - 20$  is compact, regular and oscillatory (see figure 1)

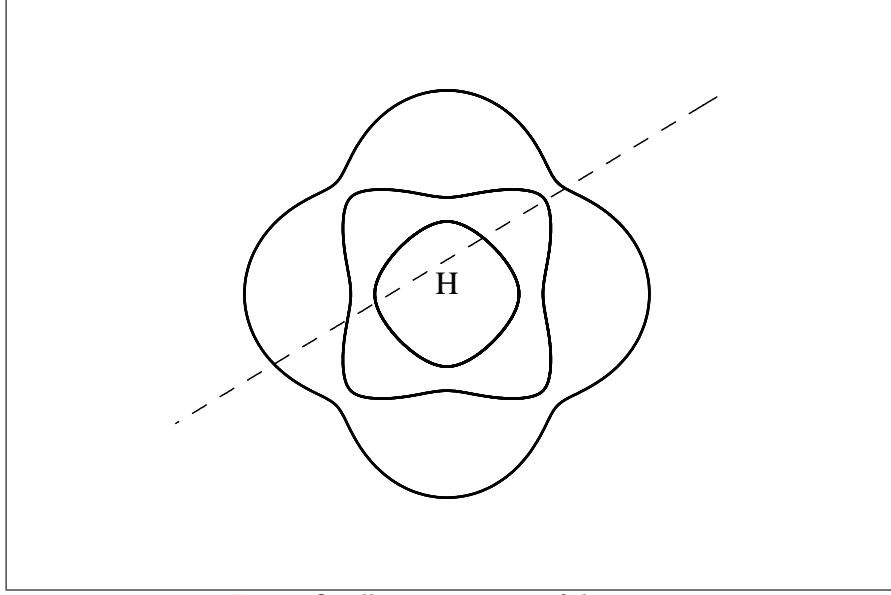


Fig. 1 Oscillatory zero set of degree 6

### 3 Photoacoustic inversion for oscillatory acquisition geometry

Consider the spherical mean transform in an Euclidean space  $E^n$  with a central set  $Z \subset E^n$ .

$$R_Z f(r, \xi) = \int_{|x-\xi|=r} f(x) dS, \quad \xi \in Z, \quad r > 0$$

**Theorem 2** *Let  $p$  be a polynomial in  $E^n$  with a compact regular oscillatory zero set  $Z$  and  $H$  be the hyperbolic cavity. An arbitrary function  $f$  in  $E^n$  with support in  $H$  can be reconstructed from its spherical means by*

$$f(x) = -\frac{p(x)}{j^{n-1}} \int_Z \left( \frac{1}{r} \frac{\partial}{\partial r} \right)^{n-1} \frac{Rf(r, \xi)}{r} \Big|_{r=|x-\xi|} \frac{d\xi}{dp} \quad (2)$$

for odd  $n$ , and

$$f(x) = -\frac{2p(x)}{j^n} \int_Z \int_0^\infty \frac{dr^2}{|x-\xi|^2 - r^2} \left( \frac{1}{r} \frac{\partial}{\partial r} \right)^{n-1} \frac{Rf(r, \xi)}{r} \frac{d\xi}{dp} \quad (3)$$

for even  $n$ , where  $Z$  is oriented by the outward conormal.

A proof is given in §6.

Explicit reconstructions of filtered back projection type are due to Finch *et al* for the case  $Z$  is a sphere [6],[7]; Xu and Wang [16] obtained explicit reconstruction when  $Z$  is a sphere, a circular cylinder and a slab. Kunyanski [10] considered some polygonal curves and polyhedral surfaces as center sets. Recently a reconstruction of FBP type was done in [12] for ellipsoids  $Z$  in 3D, and in [14] for arbitrary dimension, see also [9]. The case of variable sound velocity is addressed in [1].

**Example 3.** Half-space. The central set  $Z = \{x_1 = 0\}$  in  $E^n$ ,  $n \geq 2$  is oscillatory with two hyperbolic cavities. Such acquisition geometry appears in geophysics and in the SAR technic. Reconstructions (2) and (3) for this case look different from that of [5] and [2]. However,  $Z$  is not compact and a regularization is necessary for application of either formula.

**4.** Half-ellipsoid. Let  $p_2$  be a second order polynomial with positive principle part. The set  $p_3 = 0$  where  $p_3 = x_1 p_2$  is the union of an ellipsoid and a hyperplane. If the set  $H = \{p_2(x) < 0, x_1 > 0\}$  is not empty it is a hyperbolic cavity for  $p_3$ . We can approximate  $p_3$  by a polynomial  $\tilde{p}_3$  close to  $p_3$  with a regular oscillatory zero set  $\tilde{Z}$  (see §9). The set  $\tilde{Z}$  consists of an oval  $Z_1 = \partial H$  and an unbounded component  $Z_2$  (see figure 2):

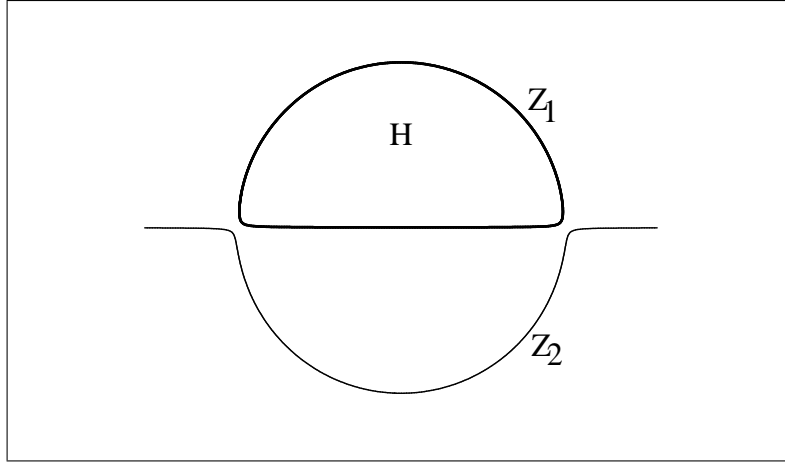


Fig. 2 Hyperbolic camera of a curve of degree 3

Note that contribution of points  $\xi \in Z_2$  to (3) decreases fast as  $\xi \rightarrow \infty$ .

## 4 Time reversal structure

Let  $E^n$  be an Euclidean space; consider the Cauchy problem for the wave equation in the space-time  $\mathbb{R} \times E^n$

$$\begin{aligned} \left( \frac{\partial^2}{\partial t^2} - \Delta \right) u &= 0 \\ u(0, x) &= 0, \quad u'_t(0, x) = f(x). \end{aligned} \tag{4}$$

for a function  $f$  in  $E^n$ . Let  $E_{n+1}(t, x)$  be the forward propagator for (4).

**Theorem 3** *Formulae (2) and (3) are equivalent to the time reversal method for (4) acting in the following steps:*

(i) *transmission (forward propagation) of a function  $f$  supported in  $H$  to the mirror manifold  $\mathbb{R} \times Z$ ,*

$$f \mapsto u(t, \xi) = \int_H E_{n+1}(t, x - \xi) f(x) dx, \quad \xi \in Z,$$

(ii) *filtration*

$$vt = Fu \doteq -\frac{\partial}{\partial t} \frac{2}{t} \frac{\partial}{\partial t} u,$$

(iii) *time reversion and retransmission*

$$g(x) = \int_Z \int_0^\infty E_{n+1}(t, x - \xi) v(-t, \xi) dt \frac{d\xi}{dp},$$

(iv) *reconstruction*

$$f(x) = -2p(x) g(x).$$

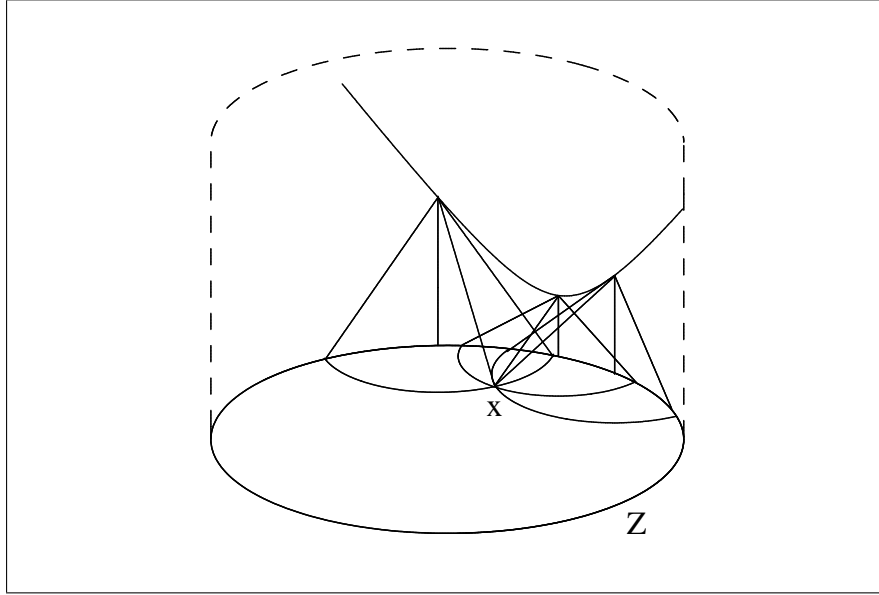


Fig. 3 Geometry of the time reversal

**Remark.** The filtration operator  $F$  is a positive self-adjoint differential operator in the Hilbert space  $L_2(\mathbb{R}_+)$ . We can replace the volume form  $d\xi/dp$  by  $q d\xi/dp$  where  $q$  is an arbitrary polynomial of degree  $\leq m - 2$  preserving equations (2) and (3). If  $q$  is a strict separator of  $Z$  (§8), the form  $q dx/dp$

is a volume form in  $Z$ . Then the retransmitting operator  $E^*$  is adjoint to the transmitting operator  $E : L_2(H) \rightarrow L_2(\mathbb{R} \times Z)$  where the space  $\mathbb{R} \times Z$  is endowed with the volume form  $qdx/dp$ . The time reversal operator  $T$  can be written in the self-adjoint form

$$T = E^*FE.$$

**Proof.** The forward propagators are

$$\begin{aligned} E_{n+1}(t, x) &= \frac{1}{2\pi} \left( \frac{\partial}{\pi \partial t^2} \right)^{(n-3)/2} \theta(t) \delta(t^2 - |x|^2) \quad \text{for odd } n, \\ E_{n+1}(t, x) &= \frac{1}{2\pi} \left( \frac{\partial}{\pi \partial t^2} \right)^{(n-2)/2} \frac{\theta(t - |x|)}{(t^2 - |x|^2)^{1/2}} \quad \text{for even } n. \end{aligned}$$

Let  $n$  be **odd**. The solution of (4) equals

$$u(t, \xi) = \frac{1}{4\pi} \left( \frac{\partial}{\pi \partial t^2} \right)^{(n-3)/2} \frac{1}{t} \int_{|x-\xi|=t} f dS = \frac{1}{4\pi} \left( \frac{\partial}{\pi \partial t^2} \right)^{(n-3)/2} \frac{Rf(t, \xi)}{t}$$

and by (2)

$$\begin{aligned} -\frac{f(x)}{2p(x)} &= \frac{2^{n-2}}{j^{n-1}} \int_Z \frac{d\xi}{dp} \left( \frac{\partial}{\partial t^2} \right)^{n-1} \frac{Rf(t, \xi)}{t} \Big|_{t=|x-\xi|} \\ &= (-1)^{(n-1)/2} \frac{1}{2\pi} \int_Z \frac{d\xi}{dp} \left( \frac{\partial}{\pi \partial t^2} \right)^{(n-3)/2} \frac{1}{t} \frac{\partial}{\partial t} \frac{1}{t} \frac{\partial}{\partial t} u(t, \xi) \Big|_{t=|x-\xi|} \\ &= -\frac{1}{\pi} \int_Z \int_0^\infty \left( \frac{\partial}{\pi \partial t^2} \right)^{(n-3)/2} \delta(t^2 - |x-\xi|^2) \frac{\partial}{\partial t} \frac{1}{t} \frac{\partial}{\partial t} u(t, \xi) dt \frac{d\xi}{dp} \\ &= \int_Z \int_{\mathbb{R}} (E_{n+1}(t, x - \xi) * v(-t, \xi)) dt \frac{d\xi}{dp} \\ &= \int_Z \int_{\mathbb{R}} (E_{n+1}(-t, x - \xi) * v(t, \xi)) dt \frac{d\xi}{dp} \end{aligned}$$

is the retransmission of  $u$ , where  $v = Fu$ .

For **even**  $n$  we consider only the case  $n = 2$ .

**Lemma 4** *We have*

$$\int_{-\infty}^{\infty} \frac{d\rho}{(\sigma - \rho)(\rho - \tau)_+^{1/2}} = \frac{\pi}{(\tau - \sigma)_+^{1/2}}$$

where  $t_{\pm} \doteq \max\{\pm t, 0\}$ .

A proof can be done by application of the Fourier transform. ►

By (3) for  $n = 2$ ,

$$-\frac{f(x)}{2p(x)} = \frac{1}{j^2} \int_Z \frac{d\xi}{dp} \int_0^\infty \frac{dr^2}{|x-\xi|^2 - r^2} \frac{1}{r} \frac{\partial}{\partial r} \frac{Rf(r, \xi)}{r}.$$

Direct propagation by Poisson's formula

$$\begin{aligned} u(t, \xi) &= \frac{1}{2\pi} \int_{|x-\xi| \leq t} \frac{f(x) dx}{(t^2 - |x - \xi|^2)^{1/2}} = \frac{1}{4\pi} \int_0^\tau \frac{d\sigma}{(\tau - r^2)^{1/2}} \frac{Rf(r, \xi)}{r} \\ &= \frac{1}{4\pi} \int_{-\infty}^\infty \frac{d\sigma}{(\tau - \sigma)_+^{1/2}} Sf(\sigma, \xi), \end{aligned}$$

where  $\tau = t^2$ ,  $r = |x - \xi|^2$ ,  $Sf(\sigma, \xi) = r^{-1} Rf(r, \xi)$  for  $\sigma = r^2$  and  $Sf(\sigma, \xi) = 0$  for  $\sigma < 0$ . Solving by Abel's method we find

$$Sf(\sigma, \xi) = 4 \frac{d}{d\rho} \int_{-\infty}^\sigma \frac{u(t, \xi)}{(\sigma - \tau)^{1/2}} d\tau,$$

where  $u(t, \xi) = 0$  for  $\tau < 0$ . We set  $\rho = r^2$  in (3) and by Lemma 4 below

$$\begin{aligned} -\frac{f(x)}{2p(x)} &= -\frac{1}{2\pi^2} \int_{\mathbb{Z}} \frac{d\xi}{dp} \int_{\mathbb{R}} \frac{d\rho}{\sigma - \rho} \frac{\partial}{\partial \rho} Sf(\rho, \xi) \\ \int_{\mathbb{R}} \frac{d\rho}{\sigma - \rho} \frac{\partial}{\partial \rho} Sf(\rho, \xi) &= \int_{\mathbb{R}} u(t, \xi) d\tau \frac{d\rho}{\sigma - \rho} \left( \frac{\partial}{\partial \rho} \right)^2 \frac{1}{(\rho - \tau)_+^{1/2}} \\ &= \int_{\mathbb{R}} \left( \frac{\partial}{\partial \sigma} \right)^2 \frac{u(t, \xi) d\tau}{(\tau - \sigma)_+^{1/2}} = \int_{\mathbb{R}} \left( \frac{\partial}{\partial \tau} \right)^2 u(t, \xi) \frac{d\tau}{(\tau - \sigma)_+^{1/2}} \\ &= \int_{\mathbb{R}} \frac{\partial}{\partial t} \frac{2}{t} \frac{\partial}{\partial t} u(t, \xi) \frac{dt}{(\tau - \sigma)_+^{1/2}} = - \int_{\mathbb{R}} \frac{v(t, \xi) dt}{(t^2 - |x - \xi|^2)^{1/2}}. \end{aligned}$$

since  $\tau = t^2$ ,  $d\tau = 2tdt$ ,  $\partial/\partial\tau = 1/2t \partial/\partial t$ . This yields

$$\begin{aligned} -\frac{f(x)}{2p(x)} &= \frac{1}{2\pi} \int_{\mathbb{Z}} \frac{d\xi}{dp} \int_{\mathbb{R}} \frac{v(t, \xi) dt}{(t^2 - |x - \xi|^2)^{1/2}} \\ &= \int_{\mathbb{Z}} E_3(-t, x - \xi) v(t, \xi) \frac{d\xi}{dp}, \end{aligned}$$

which completes the proof. ►

## 5 Proof of the reconstruction

**Proof.** Check that the generating function  $\Phi(x; \lambda, \xi) = \theta(x, \xi) - \lambda$ ,  $\theta = |x - \xi|^2$  defined in  $H \times \Sigma$ ,  $\Sigma = \mathbb{R} \times \mathbb{Z}$  satisfies conditions of Theorem 3.1 of [14]. Condition (i) is easy to check. To prove (ii) we suppose that  $y \neq x$  are conjugate points in  $H$ , which means

$$\theta(x, \xi) = \theta(y, \xi), \quad d_\xi \theta(x, \xi) = d_\xi \theta(y, \xi) \quad (5)$$

for some  $\xi \in Z$ . The first equation (5) implies that  $|x - \xi| = |y - \xi|$ . It follows that the line  $L$  through  $s = 1/2(x + y)$  and  $\xi$  is orthogonal to  $x - y$ . By the second condition (5) we have  $\langle x - y, d\xi \rangle$ , hence vector  $x - y$  is orthogonal to  $Z$  at  $\xi$ . Therefore  $L$  is tangent to  $Z$  at  $\xi$  which is impossible, since  $H$  is convex and no line through  $s \in H$  is tangent to  $Z$ . Consider an integral

$$\Theta_n(x, y) = \int_Z \frac{\omega_Z}{(\varphi(x, y; \xi) - i0)^n},$$

where

$$\varphi(x, y; \xi) = \theta(y, \xi) - \theta(x, \xi) = 2\langle x - y, \xi \rangle + |y|^2 - |x|^2 = \langle z, \xi - s \rangle, \quad z = 2(x - y).$$

**Lemma 5** *We have  $\operatorname{Re} i^n \Theta_n(x, y) = 0$  for arbitrary  $y \neq x$ .*

**Proof.** Let  $S$  be the unit sphere in  $E^n$ . We can write  $dp = p'_t dt + d_\omega p$  by means of spherical coordinates  $\xi = a + t\omega$ ,  $t \in \mathbb{R}$ ,  $\omega \in S$ . This yields

$$d\xi = t^{n-1} dt \wedge \Omega = t^{n-1} \frac{dp}{p'_t} \wedge \Omega, \quad \frac{d\xi}{dp} = \frac{t^{n-1}}{p'_t} \Omega, \quad (6)$$

where  $\Omega$  is the volume form in the sphere  $S$ . We move the origin to the point  $s \in H$  and have  $\varphi(\xi) = \langle z, \xi \rangle$ . Let  $t_{-\mu} < \dots < t_{-1} < 0 < t_1 < \dots < t_\mu$  be all zeros of  $p(a + t\omega)$  as in Theorem 1. For **even**  $n$ , we have

$$2 \operatorname{Re} \Theta_n(x, y) = 2 \int_Z \frac{1}{\varphi^n} \frac{d\xi}{dp} = \lim_{\varepsilon \rightarrow 0+} \int_Z \left[ \frac{1}{\varphi(t\omega_+)^n} + \frac{1}{\varphi(t\omega_-)^n} \right] \frac{d\xi}{dp}, \quad (7)$$

where  $\omega_\pm = \omega \pm i\varepsilon|z|^{-2}z$ ,  $\varepsilon > 0$  is a small number. We have  $\varphi(t\omega_+) = \langle tz, \omega_\pm \rangle = \langle tz, \omega \rangle \pm i\varepsilon \operatorname{sgn} t$  and

$$\frac{1}{\varphi(t\omega_+)^n} + \frac{1}{\varphi(t\omega_-)^n} = \frac{1}{t^n} \left( \frac{1}{\langle z, \omega_+ \rangle^n} + \frac{1}{\langle z, \omega_- \rangle^n} \right)$$

for any  $t \neq 0$ . Taking in account (6), we integrate over  $Z$  and get

$$2 \operatorname{Re} \Theta_n(x, y) = \lim_{\varepsilon \rightarrow 0} \int_{S_+} \sum_{k=-\mu}^{\mu} \frac{1}{t_k p'_t(a + t_k \omega)} \left[ \frac{1}{\langle z, \omega_+ \rangle^n} + \frac{1}{\langle z, \omega_- \rangle^n} \right] \Omega, \quad (8)$$

where  $S_+$  is an arbitrary hemisphere. The sum in (8) is equal to the sum of residues  $\operatorname{res}_{t_k} \rho(t, \omega)$  of the form  $\rho(t, \omega) = dt/p(a + t\omega)t$ . Integrate this form along a circle of radius  $R > \max_S |t_\mu(\omega)|$  and apply the Residue theorem:

$$\begin{aligned} \frac{1}{2\pi i} \int_{|t|=R} \rho(t, \omega) &= \operatorname{res}_0 \rho(t, \omega) + \sum_{k=-\mu}^{\mu} \operatorname{res}_{t_k} \rho(t, \omega) = \frac{1}{p(a)} + \sum_{k=-\mu}^{\mu} \frac{1}{t_k p'_t(t_k \omega)} \\ &= -\operatorname{res}_\infty \rho(t, \omega) = 0. \end{aligned}$$



Here  $t_k \omega \in Z_{|k|}$ ,  $k = -\mu, \dots, \mu$  and the residue at infinity vanishes since  $\rho(t, \omega) = O(t^{-2})$ . Therefore

$$\sum_{k=-\mu}^{\mu} \frac{1}{t_k p'_t(a + t_k \omega)} = -\frac{1}{p(a)} \quad (9)$$

and we come up with the equation

$$\begin{aligned} 2 \operatorname{Re} \Theta_n(x, y) &= -\frac{1}{p(a)} \lim_{\varepsilon \rightarrow 0} \int_{S_+} \left( \frac{1}{\langle z, \omega_+ \rangle^n} + \frac{1}{\langle z, \omega_- \rangle^n} \right) \Omega \\ &= -\frac{1}{2p(a)} \operatorname{Re} \int_S \frac{\Omega}{(\langle z, \omega \rangle - i0)^n}. \end{aligned}$$

The right hand side vanishes by [14] Proposition 4.3 hence  $\operatorname{Re} \Theta_n(x, y) = 0$ .

For **odd**  $n$ , we argue in the similar way:

$$\begin{aligned} 2i \operatorname{Im} \Theta_n(x, y) &= \lim_{\varepsilon \rightarrow 0} \sum_{k=-\mu}^{\mu} \int_{S_+} \frac{1}{t_k p'_t(a + t_k \omega)} \left[ \frac{1}{\langle z, \omega_- \rangle^n} - \frac{1}{\langle z, \omega_+ \rangle^n} \right] \Omega \\ &= -\frac{2i}{p(a)} \operatorname{Im} \int_{S^{n-1}} \frac{\Omega}{(\langle z, \omega \rangle - i0)^n}. \end{aligned} \quad (10)$$

The right hand side vanishes according to [14] Proposition 4.3. This together with (10) implies vanishing of  $\operatorname{Im} \Theta_n(x, y)$  and completes the proof of Lemma 5.  $\blacktriangleright$

Theorem 2 now follows from [14] Theorem 3.1 applied for the generating function  $\Phi$  as above and for the space  $\Sigma = \mathbb{R} \times Z$  endowed with the form  $dx/dp$ . We change the variable  $\lambda = r^2$  and take into account that  $|\nabla \theta| = 2|x - \xi| = 2r$  and  $Mf(r^2, x) = (2r)^{-1} Rf(r, x)$  in loc. cit. To complete the proof we only need to calculate the dominator

$$D_n(x) = \frac{1}{|S^{n-1}|} \int_Z \frac{1}{|\xi - x|} \frac{d\xi}{dp(\xi)}$$

for an arbitrary  $x \in H$ . For any  $\xi \in Z$ , we can write  $\xi = x + t_k(\omega)\omega$  for a unique  $\omega \in S$  and  $t_k > 0$  and have  $|\xi - x| = t_k(\omega)$ . Replacing  $a$  by  $x$  in (6) yields

$$D_n(x) = \frac{1}{|S^{n-1}|} \int_S \sum_{k=1}^{\mu} \frac{\Omega}{t_k p'_t(x + t_k(\omega)\omega)}.$$

The sum of contributions of opposite points  $\omega \in S_+$  and  $-\omega$  equals

$$\begin{aligned} &\sum_{k=1}^{\mu} \frac{\Omega}{t_k(\omega) p'_t(x + t_k(\omega)\omega)} - \sum_{k=1}^{\mu} \frac{\Omega}{t_k(-\omega) p'_s(x - t_k(-\omega)\omega)} \\ &= \sum_{k=-\mu}^{\mu} \frac{\Omega}{t_k(\omega) p'_t(x + t_k\omega)}, \end{aligned}$$

where  $s = -t$  and  $t_k(-\omega) = -t_{-k}(\omega)$ ,  $k = 1, \dots, \mu$ . By (9) this sum is equal to  $-1/p(x)$ , hence

$$D_n(x) = -\frac{1}{|S^{n-1}|p(x)} \int_{S_+} \Omega = -\frac{1}{2p(x)},$$

which completes the proof of (2) and (3). ►

## 6 Separators

**Definition.** Let  $p$  be an oscillatory polynomial of degree  $m > 2$  with a hyperbolic point  $a$ . We say that a polynomial  $q$  *separates*  $p$  from  $a$ , if for almost any line  $L$  through  $a$ , each interval between consecutive zeros of  $p$  on  $L$  contains just one zero of  $q$ , except for the interval that contains  $a$ , where  $q$  does not vanish (see figure 4). It follows that  $q$  has, at least,  $m - 2$  zero on  $L$ , hence  $\deg q \geq m - 2$ . We say that a separator  $q$  of a polynomial  $p$  is *strict* if  $\deg q = m - 2$ .

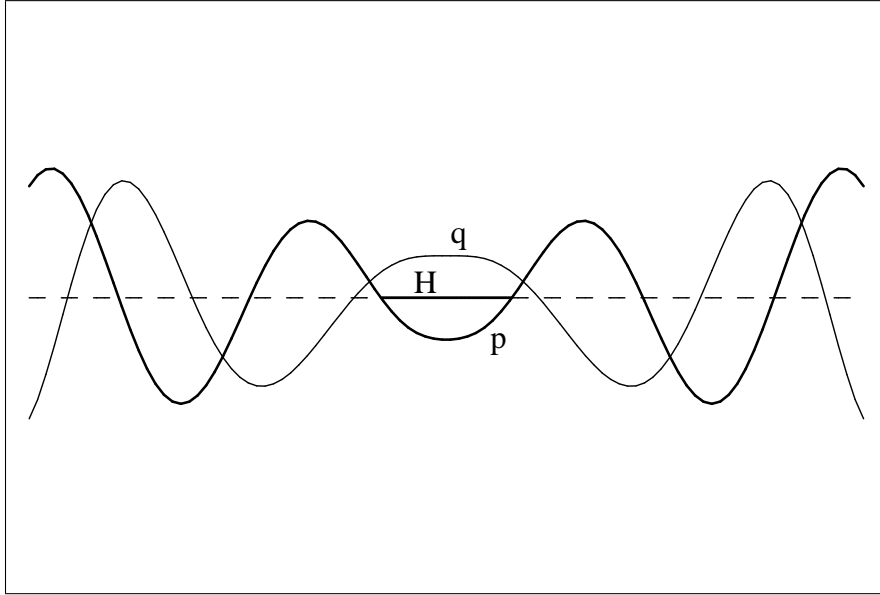


Fig. 4 Polynomial  $p$  and a separator  $q$

One can find a separator  $q$  by the following method.

**Theorem 6** *For an arbitrary compact oscillatory set  $Z$  of degree  $m$ , there exists a polynomial  $q$  of degree  $< m$  that separates  $Z$  from any hyperbolic point.*

**Proof.** The case  $n = 1$  is trivial; we assume that  $n > 1$ . Take a hyperbolic point  $a$  and consider the Euler field  $\mathbf{e}_a = \sum (x_i - a_i) \partial / \partial x_i$  centered at  $a$ . The polynomial  $q_a \doteq \mathbf{e}_a(p) - mp$  has degree  $\leq m - 1$  and for any unit  $\omega$ ,

$$q_a(a + t\omega) = t^{m+1} \frac{d}{dt} (t^{-m} p(a + t\omega)) = tp'_t(a + t\omega) - mp(a + t\omega). \quad (11)$$

We numerate roots of  $p(a + t\omega)$  by  $t_k = t_k(\omega)$ ,  $k = \pm 1, \dots, \pm \mu$  as in (1) so that  $t_k$  has the same sign as  $k$ . By (11) and Rolle's theorem  $q_a(a + t\omega)$  has, at least,  $m - 2$  roots  $s_k = s_k(\omega)$ ,  $k = \pm 1, \dots, \pm(\mu - 1)$  such that

$$t_k \leq s_k \leq t_{k+1}, \quad t_{-k-1} \leq s_{-k} \leq t_{-k}, \quad k = 1, \dots, \mu - 1. \quad (12)$$

By continuity it is true for all unit  $\omega$  occasionally with non strict inequalities. Check that there are exactly  $m - 2$  such zeros  $s_k$  counting with multiplicities. For any  $\omega \in S$  except for a set of zero measure all zeros  $t_k$  are simple and all inequalities (12) are strict. Suppose that one of the intervals (12) say  $I_k \doteq (t_k, t_{k+1})$  contains more than one zero  $s_k$ . The total number  $r_k$  of zeros of  $q_a$  in  $I_k$  is odd, since the polynomial  $t^{-m}p(a + t\omega)$  does not vanish in  $I_k$ . It follows that  $r_k \geq 3$  and  $r_i \geq 1$  for any  $i$ , which implies  $\sum_i r_i \geq m$ . This is not possible since  $\deg q_a \leq m - 1$ . By continuity (12) holds also for any unit  $\omega$  in the sense that  $q_a(a + s\omega)$  has a zero  $s$  of multiplicity  $m - 1$ , if this point is a root of  $p(a + s\omega)$  of multiplicity  $m$ . It implies that all zeros  $s = s_k(\omega)$ ,  $k = 1, \dots, \mu - 1$  are unambiguously defined and by Rouché's theorem are continuous functions of  $\omega$ . We have  $s_k(-\omega) = -s_{-k}(\omega)$  and each variety

$$W_k = \{x = a + s_k(\omega)\omega, \quad \omega \in S\}, \quad k = 1, \dots, \mu - 1$$

is closed and homeomorphic to a sphere. By (12) these varieties separate hypersurfaces  $Z_k, k = 1, \dots, \mu$  constructed in Theorem 1. Check that  $q(a + t\omega)$  does not vanish for  $t \in (t_{-1}, t_1)$ . We can assume that  $p < 0$  in  $H$  and have  $q_a(a + t_{\pm 1}\omega) = t_{\pm 1}p'_t(a + t_{\pm 1}\omega) \geq 0$ . If  $q$  vanishes in a point  $s \in (t_{-1}, t_1)$ , it must have  $\geq 2$  zeros, which is impossible since  $\deg q_a < m$ . This shows that  $q_a$  separates  $p$  from  $a$ . It follows that  $q_a$  separates  $p$  from any other point  $b \in H$  since the hyperbolic cavity  $H$  is inside of all ovals  $W_k$ . ►

**Corollary 7** *Any separator  $q$  of a compact oscillatory set  $Z$  of degree  $m = 2\mu$  is an oscillatory polynomial with a hyperbolic cavity  $G \supset H$ . The zero set  $W$  of  $q$  is the union of continuous ovals  $W_1, \dots, W_{\mu-1}$  and of a closed unbounded component  $W_\mu$  if  $q$  is not strict such that the sets  $H, Z_1, W_1, Z_2, \dots, Z_{\mu-1}, W_{\mu-1}, Z_\mu, W_\mu$  are nested (see figure 1).*

**Proof.** Any strict separator has no zeros  $x = a + s\omega$  except for  $s = s_k$  as in (12). If  $q$  is not strict it must have exactly one such real zero, say  $s = s_\mu(\omega)$ , which is defined and continuous for  $\omega \in S \setminus S'$ , where  $S'$  is a subset of dimension  $< n - 1$ . This function is odd, since any line  $L_\omega = \{x = a + s\omega, \quad s \in \mathbb{R}\}$ ,  $\omega \in S \setminus S'$  contains only one such zero. It follows that  $S'$  divides the unit sphere in two opposite parts  $S_\pm$  and  $s_\mu(\omega) \rightarrow \infty$  as  $\omega \rightarrow S'$ . The equation  $x = a + s_\mu(\omega)\omega$  defines an unbounded component  $W_\mu$  for  $\omega \in S_+$ . ►

**Proposition 8** *Let  $p$  be an oscillatory polynomial of degree  $m$  such that the Taylor series of  $p(a + y)$  does not contain terms of degree  $m - 1$  for a hyperbolic point  $a$ . Then  $q_a = \mathbf{e}_a(p) - mp$  is a strict separator of  $p$ . In particular, for any even polynomial  $p$  with compact zero set the polynomial  $q_0$  is a strict separator.*

**Proof.** The polynomial  $q_a$  is a strict separator since  $\deg q_a \leq m - 2$ . If  $p$  is even and  $Z$  is compact, the unique hyperbolic cavity  $H$  is symmetric with respect to the origin and contains the origin since it is convex. Therefore the second statement follows from the first one. ►

Note without proofs few more geometric properties of oscillatory sets.

**Proposition 9** *Any compact oscillatory set  $Z$  that has a strict separator  $q$  can be approximated by regular oscillatory sets  $\tilde{Z}$  whose strict separators approximate  $q$ .*

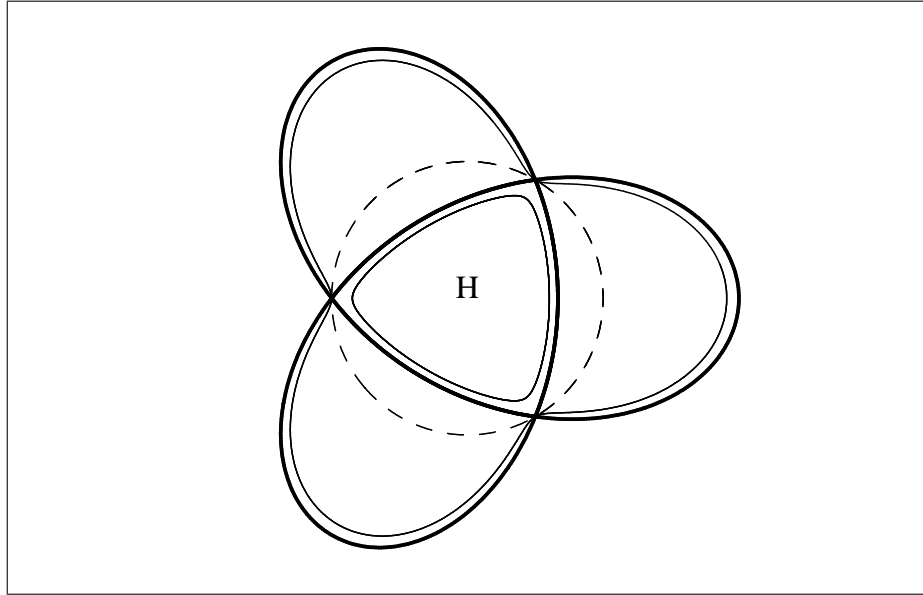


Fig. 5 Hypotrochoid (thick) and a regular approximation (thin)

**Example 5.** A hypotrochoid given by  $p(x, y) = 4(x^2 + y^2)^2 - 4x^3 + 12xy^2 - 27(x^2 + y^2) + 27$  is oscillatory and  $q = 4(x^2 + y^2) - 9$  is a strict separator (see figure 5).

**Proposition 10** *Any compact convex domain  $Q$  in  $\mathbb{R}^n$  can be approximated by regular hyperbolic cavities  $H$  of oscillatory sets admitting strict separators.*

## 7 A generalization of Newton's levitation theorem

I. Newton [13] proved that a mass uniformly distributed over a thin sphere  $S \subset E^3$  generates the zero gravitation field inside the sphere. The same true for the cavity of a solid layer between two ellipsoids homothetic with respect to the center. P. Dive [4] called a monoid the layer between any two closed homothetic

surfaces with respect to an interior point. He proved that any non-ellipsoidal monoid can not create levitation in the cavity. V. Arnold [3] constructed a distribution of electric charge on a regular compact oscillatory set  $Z$  that generates the zero electrical field in the hyperbolic cavity  $H$ . This distribution, however, has variable sign except if  $Z$  is not an ellipsoid. Similar problems for magnetic fields were studied in [15]. We show that for any oscillatory set  $Z$  that admits a strict separator, there exists a strictly positive mass distribution on  $Z$  that generates levitation in  $H$ . If  $Z$  is regular, the set of such mass distributions form a convex cone of dimension  $\left\lfloor \frac{m-2+n}{n} \right\rfloor$ . Also layers bounded by close level sets of the corresponding oscillatory polynomial  $p$  generate levitation in a cavity. The key notion is a strict separator of an oscillatory polynomial.

**Theorem 11** *For an arbitrary oscillatory polynomial  $p$  in  $E^3$  with compact zero set and any strict separator  $q$ , the distribution of mass in  $E^3$  with density  $|q| \delta(p)$  generates the zero gravitation field in the hyperbolic cavity  $H$  of  $p$ .*

**Proof.** For any  $\omega \in S \setminus S'$ , the polynomial  $p_\omega(t) = p(a + t\omega)$  has  $m$  real zeros  $t_k(\omega)$ ,  $k = \pm 1, \pm 2, \dots, \pm \mu$  numerated as in Theorem 1. By the Residue theorem we have

$$\sum \frac{q(a + t_k \omega)}{p'_t(a + t_k \omega)} = \sum \operatorname{res}_{t_k} \frac{q(a + t\omega)}{p'_t(a + t\omega)} dt = -\operatorname{res}_\infty \frac{q}{p} dt = 0. \quad (13)$$

According to the numeration a zero  $t_k$  is positive or negative together with  $k$ . Suppose that  $p < 0$  and  $q > 0$  in  $H$ . Signs of  $p'_t(a + t_k \omega)$ ,  $k = -\mu, \dots, -1, 1, \dots, \mu$  alternate and  $kp'_t(a + t_k \omega) > 0$  for any odd  $k$  whereas  $kp'_t(a + t_k \omega) < 0$  with even  $k$ . The sign of  $q(a + t_k \omega)$  alternates in the different way:  $q(a + t_k \omega) > 0$  for odd  $k$  and  $q(a + t_k \omega) < 0$  for even  $k$ . Therefore we have

$$\frac{q(a + t_k \omega)}{p'_t(a + t_k \omega)} > 0 \text{ for } k > 0; \quad \frac{q(a + t_k \omega)}{p'_t(a + t_k \omega)} < 0 \text{ for } k < 0. \quad (14)$$

The sum of these fractions vanishes by (13). By Newton's law the gravitation field decreases at the same rate as a beam of straight rays diverges. Following Newton's geometrical method we consider a small solid angle  $C(a, \omega) \subset E$  with vertex at  $a$  of spherical measure  $d\omega$ . The mass of a piece of  $Z \cap C(a, \omega)$  at a point  $x = a + t_k \omega$  is equal to

$$m_k \doteq |t_k|^2 \frac{|q(x)|}{|\langle \omega, \nabla p(x) \rangle|} \Omega = |t_k|^2 \left| \frac{q(x)}{p'_t(x)} \right| \Omega.$$

Its contribution to the field at  $a$  is equal to  $F_k \doteq m_k |t_k|^{-2} d\omega$  for  $k > 0$  and  $F_k \doteq -|t_k|^{-2} m_k \Omega$  for  $k < 0$ , since these points are on opposite site of  $a$ . By (14) the total of these contributions equals the sum (13) times  $\Omega$ , hence cancels. Theorem, follows since this conclusion holds for almost all  $\omega$ . ►

**Remark 1.** For the polynomial  $p = |x|^2 - 1$ , the above construction gives  $q_0 = 2$ . Theorem 11 guarantees levitation in a sphere generated by a uniform distribution of mass on the sphere. This is Newton's attraction theorem. If

$Z$  is a compact regular oscillatory set of degree  $m$  and  $q$  is a strict separator, then any polynomial  $\tilde{q}$  of degree  $m - 2$  that is sufficiently close to  $q$  is a strict separator. By Theorem 11 for any such  $\tilde{q}$ ,  $\tilde{q}d\xi/dp$  is a volume form in  $Z$ . Any mass distribution in  $Z$  that is proportional this form admits levitation in  $H$ .

**Remark 2.** Theorem 11 is generalized for arbitrary linear space  $\mathbb{R}^n$ , if the corresponding "gravitation" force generated by a delta-like mass at the origin has the potential  $U = \sigma(\omega) r^{-n+1}$ , where  $\sigma(\omega)$  is an arbitrary even function of  $\omega \in S$ .

**Corollary 12** *Under conditions of Theorem 11, if  $p < 0$  in  $H$ , the distribution of mass with density  $|q| dx$  in the layer  $L \doteq \{x : a \leq p(x) \leq b\}$  generates the zero gravitation field in  $H$  for arbitrary  $a, b$  such that  $q \neq 0$  in  $L$ .*

**Proof.** By Theorem 11, the density  $|q| \delta(p - \lambda)$  generates the zero gravity in  $H$  for any  $\lambda$  such that  $q$  separates  $p - \lambda$ . By Fubini's the same is true for the density

$$\int_a^b |q| \delta(p - \lambda) d\lambda = |q| dx$$

supported by the layer  $\{a \leq p \leq b\}$  if  $q$  separates  $p - \lambda$  for  $a \leq \lambda \leq b$ . ►

A layer generated by a hypotrochoid is shown in figure 5.

**Example 6.** A surface of normals of the system of crystal optics is given by the equation  $p(\xi) = 0$  where

$$\begin{aligned} p(\xi) = & (\sigma_1 \xi_1^2 + \sigma_2 \xi_2^2 + \sigma_3 \xi_3^2) |\xi|^2 \\ & - (\sigma_3 + \sigma_2) \sigma_1 \xi_1^2 - (\sigma_1 + \sigma_3) \sigma_2 \xi_2^2 - (\sigma_1 + \sigma_2) \sigma_3 \xi_3^2 + \sigma_1 \sigma_2 \sigma_3 \end{aligned}$$

is an even elliptic oscillatory polynomial. The polynomial  $q = \mathbf{e}_0(p) - 4p$  is a strict separator.

## 8 Non strict case

**Theorem 13** *Let  $p$  be an elliptic oscillatory polynomial of degree  $m$  and  $q$  be a separator. The gravitation field generated by mass distribution in  $Z$  with the density  $|q| \delta(p)$  is constant in the hyperbolic cavity and equals*

$$F = - \int_{\mathbb{P}^{n-1}} \frac{q_{m-1}(\omega)}{p_m(\omega)} \cdot \omega \, \Omega, \quad (15)$$

where integration can be taken over an arbitrary unit hemisphere.

Note that the integrand  $q_{m-1}(\omega) / p_m(\omega) \cdot \omega$  is an even vector function of  $\omega$ .

**Proof.** Let again  $C(a, \omega)$  be a small solid angle as in Theorem 11. By (13) the gravitation field generated by points on  $Z \cap C(a, \omega)$  upon a point  $a \in H$  equals the vector

$$\sum_{k=1}^m \frac{q(x)}{p'_k(x)} \cdot \omega \, \Omega.$$

where  $x = a + t_k \omega$ . By the Residue theorem, the sum of scalars is equal to

$$\sum_{k=1}^m \frac{q(x)}{p'_t(x)} = -\operatorname{res}_\infty \frac{q}{p} dt = -\frac{q_{m-1}(\omega)}{p_m(\omega)}.$$

Integrating over any unit hemisphere, we get (15). ►

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